Section-I: (1) If (a_n) and (b_n) are such that $|a_n - b_n| \to 0$ and $a_n \to a$, then $b_n \to a$.
(2)Suppose that $\sum a_n$ converges and $a_n \geq 0$. Prove that $\sum \sqrt{a_n a_{n+1}}$ converges.
(3)Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and $f'(x) \neq 0$ for all x. Then f is one-one.

Solution: (i) The answer follows from the inquality $|b_n - a| \leq |b_n - a_n| + |a_n - a|$. (ii) Use the AM-GM inequality directly to prove the result. (iii) If f is not one-one, then we have $f(a) = f(b)$ for some $a < b$. By the mean value theorem, the derivative vanishes at some point in (a, b) which contradicts the given condition.

□

Section-II: Answer any 4 questions, 6 marks each. (1) Let (a_n) be a sequence. Prove that limsupa_n is a limit point of $\{a_n\}$.
(2) If $a_n > 0$ and $\sum_n a_n$ converges, then prove that $\sum_n a_n/s_n$ diverges, where $s_n = \sum_{k=n}^{\infty} a_k$.
(3) Let $f : [a, b] \to \mathbb{R}$ be continuous. If f has no local minima or local maxima at any point in (a, b), then f is monotonic.
(4) Prove that monotone functions do not have discontinuities of second kind, and the set of discontinuities is countable.
(5) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with $f'(x) = 0$, but $f''(x) < 0$ for some x. Show that f has a local maximum at x.
(6) Let $f : \mathbb{R} \to \mathbb{R}$ a differentiable function and $g(x) = f(x + 1) - f(x)$. Prove that if $f'(x) \to 0$ as x goes to ∞ or $-\infty$, then g is bounded.

Solution:(1) If α is the limsup of the sequence, then show that there exists a subsequence (n_k) such that a_{n_k} converges to the limsup. This will show that limsup is a limit point. (2) The idea is to imitate the proof the divergence of the harmonic series $\sum_n 1/n$. First step is to choose n_1 and n_2 such that $\sum_{k=n_1}^{n_2} ak/s_k > 1/2$. Now from the series $\sum_n a_n/s_n$, remove the first n_2 terms, and then, from the new series, choose n_3 and n_4 such that $\sum_{k=n_1}^{n_2} ak/s_k > 1/2$. Thus we get $\sum_n a_n/s_n > 1/2+1/2+$...... (3) Suppose that f is not monotone. There are many cases: monotically not increasiing or decreasing. We only conisder one case. Assume that there exist points $x < z < y$ such that $f(x) < f(z) > f(y)$. Since f is continuous, its image of any closed, bounded interval is of same kind. Hence, $f(z)$ is local mamimum, which condtractis the given assumption.(4) and (5) are standard facts: see Rudin's Principles of mathematical analysis or Thomas Calculus. (6) Recall the mean value theorem. Now, there exists a c_x, which depends on x, such that $|f(x + 1) - f(x)| = f'(c_x)$. As x goes to $+or - \infty$, c_x goes to $+or - \; infty$. By the assumption, we have $g(x)$ is bounded. □

Section-II: Answer any 2 questions, 10 marks each. (1)(a) If x is a limit point of (a_n), prove that $\liminf_n a_n \leq x \leq \limsup_n a_n$. (1)(b) Srt $\alpha := \limsup_n |a_n|^{1/n}$. Prove that if $\alpha < 1$, then the series converges and if $\alpha > 1$, the series diverges.
(2) Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be continuous. Show that there are x, y such that $f(x) \leq f(t) \leq f(y)$ for all $t \in I$. Further if f is one-one, then f is a homemorphism between I and $f(I)$.
(3) (a) Let f, g be differentiable functions on $\mathbb{R}$. Prove that for $r < s$, there is a t between r and s such that $[f(r) - f(s)]g'(t) = [g(r) - g(s)]f'(t)$.(b) Let $g(x) = x$. Obtain t in (a) as a function of r and s, find $\lim_{s \to r} t$ when $f(x) = x^2$ and $f(x) = x^2$.

Solution: (1) These are standard facts, see Kreyzig's Advanced Engg. Mathematics and Thomas Calculus. (2) See Rudin's book for the first part. The second part follows straifgtforward: the only thing to be proved is that f^{-1} is continuous from $f(I)$ to I. See Munkres, section on continuous functions. (3) This is the Cauchy's mean value theorem. Geometrically, this means that there is a tangent to the curve $(f(t), g(t))$ parrellel to the line given by $(f(a), g(a))$ and $f(b), g(b))$. These are standard. $\square$